# CY-OPERATORS AND L-FUNCTIONS

DUCO VAN STRATEN

Abstract. This a write up of a talk given at the MATRIX conference at Creswick in 2017 (to be precise, on Friday, January 20, 2017.) It reports on work in progress with P. Candelas and X. de la Ossa. The aim of that work is to determine, under certain conditions, the local Euler factors of the L-functions of the fibres of a family of varieties *without recourse to the equations* of the varieties in question, but solely from the associated *Picard-Fuchs equation.*

## 1. Introduction

It is very honourable to speak the last words in this nice conference; surely these words are not the last on hypergeometrics, but rather some further exploration into *Transhypergeometria*, the unknown land of our dreams. I will report on joint work in progress with Philip Candelas and Xenia de la Ossa, [9]. I will start with some motivation.

## 2. Elliptic curves versus Rigid Calabi-Yau threefolds

Elliptic curves and rigid Calabi-Yau manifolds share many common features. As a topological space, an elliptic curve is isomorphic to $S^1 \times S^1$ and a rigid Calabi-Yau threefold is a bit like $S^3 \times S^3$, at least what its third cohomology is concerned. On the arithmetic level, an elliptic curve $E$ defined over $\mathbb{Q}$ determines a two dimensional motive $H^1(E)$ and in a similar way a rigid Calabi-Yau threefold $X$ defined over $\mathbb{Q}$ produces a two dimensional motive $H^3(X)$. There are Hodge and $p$-adic realisations, giving rise to $L$-functions that come from classical modular forms for some $\Gamma_0(N)$.

| Space | Motive | Hodge | Frobenius | Weil | Hecke |
|---|---|---|---|---|---|
| $E/\mathbb{Q}$ | $H^1(E)$ | 0 1 1 0 | $T^2 - a_pT + p$ | $\lvert a_p\rvert \le p^{1/2}$ | $L(H^1(E)) = L(f),\ f \in S_2(\Gamma_0(N))$ |
| $X/\mathbb{Q}$ | $H^3(E)$ | 1 0 0 1 | $T^2 - a_pT + p^3$ | $\lvert a_p\rvert \le p^{3/2}$ | $L(H^3(X)) = L(f),\ f \in S_4(\Gamma_0(N))$ |

By the great theorem of Wiles [35], [36] we know that all elliptic curves over $\mathbb{Q}$ are modular, and by further development of these methods, it was shown that rigid Calabi-Yau threefolds defined over $\mathbb{Q}$ are also modular, [18], [14].

However, there are also big differences between these two cases. Elliptic curves depend on a single modulus and form nice families. Classical normal forms are provided by the *Legendre family*

$$L_\lambda : y^2 = x(x-1)(x-\lambda)$$

or the *Hesse family*

$$H_\lambda : x^3 + y^3 + z^3 + \lambda xyz = 0\,,$$

where $\lambda$ is the parameter.

This work is partly supported by DFG Sonderforschungsbereich/Transregio 45.

On the other hand, as by definition $h^{12}=0$, rigid Calabi-Yau spaces do not admit any non-trivial deformations, and their occurrence is sporadic. No general description or construction is known for them. We refer to [22], [37] for an overview of the exciting bestiary.

***Question***

*Which weight four cups forms appear as modular form of rigid Calabi-Yau manifolds?*

For example, as can be seen from consulting [22], there are many different rigid Calabi-Yau varieties leading to the weight four cusp form for $\Gamma_0(6)$, but I do not know of any rigid Calabi-Yau threefold realising the weight four cusp form for $\Gamma_0(7)$.

2.1. **How can rigid varieties appear in a pencil?** Let us look at an example. The famous *Schoen quintic* $X_1$ studied in [26] is the degree 5 hypersurface in $\mathbb{P}^4$ given by the equation

$$X_1 : x_1^5+x_2^5+x_3^5+x_4^5+x_5^5 = 5x_1x_2x_3x_4x_5.$$

It is easily seen to have the 125 points

$$x_i^5 = 1, \quad x_1x_2x_3x_4x_5 = 1$$

as nodal singularities. There exists a small resolution $\pi : X \longrightarrow X_1$ that replaces each node by a projective line $\mathbb{P}^1$. $X$ is a rigid Calabi-Yau threefold: the infinitesimal deformations of $X$ can be identified with the infinitesimal deformations of $X_1$ *for which the nodes lift*, which are none. For small prime numbers the Euler factors of the $L$-function can be determined counting points of $X_1$ and correcting these counts to get the numbers of points of the resolved manifold $X$. As the Galois representation is determined by finitely many Euler factors, it was found that the $L(H^3(X_1)) = L(f)$ for some $f \in S_4(\Gamma_0(25))$, which was identified by C. SCHOEN.

Now note that the quintic $X_1$ (and not $X$) is a member of the even more famous *Dwork pencil*

$$X_\psi : x_1^5+x_2^5+x_3^5+x_4^5+x_5^5 = 5\psi x_1x_2x_3x_4x_5$$

that stands at the beginning of the mirror symmetry story, for which we refer to [8], [11], [33], [24] . The third cohomology of $X_\psi$ is the direct sum of two pieces

$$H^3(X_\psi) = V \oplus F\,.$$

Here the part $F$ has Hodge numbers 0 100 100 0, and the part $V$ has Hodge numbers 1 1 1 1. The Picard-Fuchs equation for this part leads to the hypergeometric differential equation

$$\mathcal{P} := \Theta^4 - 5^5 t(\Theta+\frac{1}{5})(\Theta+\frac{2}{5})(\Theta+\frac{3}{5})(\Theta+\frac{4}{5}), \;\; t=1/(5\psi)^5, \;\; \Theta = t\frac{d}{dt}\,,$$

which describes a *variation of Hodge structures* (VHS) over $S := \mathbb{P}^1 \setminus \{0, 1/5^5, \infty\}$. At the three singular points these Hodge structures degenerate into *mixed Hodge structures* (MHS). We refer to [25] for a detailed account of (mixed) Hodge theory. Quite generally, the Jordan structure of the local monodromy determines the weight filtration. At $t=0$ we have a so-called MUM-point, the monodromy has a maximal Jordan block. The mixed Hodge diamond looks like

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

(The weight is equal to the height in the diagram, counted by putting lowest row at height zero; the operator $N$ shifts two steps downwards.) The limiting mixed Hodge structure is an iterated extension of Tate Hodge structures and it leads to the extension data described in [12] that are equivalent to the so-called *instanton numbers* computed in [8].

At $t = 1/5^5$ there is a single Jordan block of size 2 (a $C$-point in the terminology of [31]). The mixed Hodge-diamond for $H^3$ looks like:

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 0 & & 0 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}
$$

So we see that the motive $Gr_3^W H = H^3(X)$ is like that of a rigid Calabi-Yau.

There is one further possible degeneration of a $(1,1,1,1)$-VHS, that does not appear in this family, namely where there are two Jordan blocks of size 2 (a $K$-point in the terminology of [31]). The mixed Hodge diamond for $H^3$ now looks like

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 1 & & 0 & & 1 & \\
0 & & 0 & & 0 & & 0 \\
 & 1 & & 0 & & 1 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}
$$

So $Gr_2^W H^3$ is a $(1,0,1)$-Hodge structure that looks like the one appearing for $K3$-surfaces with Picard number 20.

One of the motivations to look at general motivic $(1,1,1,1)$-variations over $S = \mathbb{P}^1 \setminus \Sigma$ is the natural appearance of weight four and weight three cusp forms for $\Gamma_0(N)$ at the boundary points $\Sigma \subset \mathbb{P}^1$. Such motivic $(1,1,1,1)$-variations are expected to arise from *Calabi-Yau operators*.

## 3. Calabi-Yau operators

Calabi-Yau operators, as understood in [4] and [31], are operators 'like' $\mathcal{P}$. First of all, they are *fourth order Fuchian* differential operators

$$\mathcal{P} \in \mathbb{C}[t, \Theta], \quad \Theta = t\frac{d}{dt}$$

that are *symplectic* and have 0 as a MUM-point. If we look at it from the point of view of differential operators, it is rather easy to satisfy these conditions, for example by looking at operators of the form

$$\mathcal{P} = \Theta^2 P \Theta^2 + \Theta Q \Theta + R\,,$$

where $P, Q, R$ are any polynomials with $P(0) = 1$. In order to classify as a *Calabi-Yau operator*, one has to complement these easy conditions with further arithmetical conditions that are supposed to hold if the operator is a Picard-Fuchs operator of a 1-parameter family of Calabi-Yau varieties defined over $\mathbb{Q}$. In [4] the following *integrality conditions* were put forward and used to define Calabi-Yau operators.

I. The holomorphic solution $\phi_0(t)$ has an integral power-series expansion:

$$\phi_0(t) \in \mathbb{Z}[[t]]\,.$$

II. The $q$-coordinate has an integral power series expansion

$$q(t) \in \mathbb{Z}[[t]]\,.$$

III. The normalised instanton numbers become integral

$$n_0 := 1, \;\; n_1, \;\; n_2, \ldots, n_d, \ldots$$

after multiplication by a common denominator.

Furthermore, the case where all $n_d = 0, d \geq 1$ is considered as *trivial*, as in that case $\mathcal{P}$ is the third symmetric power of a second order operator. In fact, it is more natural to have coefficients in $\mathbb{Z}[\frac{1}{N}]$, so to allow denominators involving a finite set of *bad primes*. Currently more than 500 operators are known that seem to satisfy these three conditions (see [2], [3],[10]), but condition III is not proven to hold in a single case. The first condition should already imply that the operator is of geometric origin, see [5]. There are many examples of operators that satisfy $I$, but not $II$. In a good number of cases integrality of the $q$-coordinate have been proven [21], [13]. For some time it was expected that condition $III$ was implied by $I$ and $II$, until Michael Bogner [6] found an operator that satisfies $I$ and $II$, but for which $III$ appears to fail. There exists an unpublished paper [34] in which it is claimed that Picard-Fuchs operators coming from families of Calabi-Yau varieties indeed satisfy these three arithmetical conditions.

Of course, one can also look at differential operators of order different from four, and try to single out a particular nice sub-class of Calabi-Yau operators of arbitrary order. For an account, we refer to [6] and [7].)

A particular nice example is operator **AESZ 34**

$$\Theta^4 - t(35\,\Theta^4 + 70\,\Theta^3 + 63\,\Theta^2 + 28\,\Theta + 5) +$$
$$+ t^2(\Theta+1)(259\,\Theta^2 + 518\,\Theta + 235) - 5^3 t^3 (\Theta+1)^2(\Theta+2)^2$$

that was reported to us long ago by H. Verrill, [32]. It turned up prominently at this conference, as it is associated to the 5-fold banana Feynman graph. As such, it is part of a very nice series of Calabi-Yau operators that exist for all orders. Its *Riemann symbol* (see [19]) is

$$\left\{\begin{array}{ccccc} 0 & 1/25 & 1/9 & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 2 & 2 \end{array}\right\}.$$

and the holomorphic solution has an expansion of the form

$$\phi_0(t) = \sum_{n=0}^{\infty} A_n t^n, \quad A_n := \sum_{i+j+k+l+m=n} \left(\frac{n!}{i!j!k!l!m!}\right)^2 .$$

As for all Calabi-Yau operators, there is a unique *Frobenius basis* of solutions around 0 of the form

$$\begin{array}{rcr} \phi_0(t) & = & f_0(t) \\ \phi_1(t) & = & \log(t)\phi_0(t) + f_1(t) \\ \phi_2(t) & = & \log(t)^2\phi_0(t) + 2\log(t)\phi_1(t) + f_2(t) \\ \phi_3(t) & = & \log(t)^3\phi_0(t) + 3\log(t)^2\phi_1(t) + 3\log(t)\phi_2(t) + f_3(t) \end{array}$$

where $f_0(t) \in \mathbb{Z}[[t]]$, $f_i(t) \in t\mathbb{Q}[[t]]$ $(i = 1, 2, 3)$.

The points $1/25, 1/9, 1$ are $C$-points: there appears a single logarithm 'between' the two equal exponents. The point $\infty$ is a $K$-point: there are two logarithms, again between the two pairs of equal exponents. At each of the conifold points should appear a weight four modular form of some level, at $\infty$ there is a weight three modular form.

## 4. Euler factors from Picard-Fuchs operators

It has been known from the work of Dwork [15], [16] that there is a very tight link between the Frobenius operator and the Picard-Fuchs operator in a family of varieties. For the sake of concreteness, let us consider as before a family $Y_t$ of Calabi-Yau 3-folds defined over $\mathbb{Q}$ with a MUM-point at 0 and let us fix a prime $p$. Then the Frobenius operator

$$F := F_p \in Aut(H^3(Y_t))$$

has a characteristic polynomial $P(T) = \det(T - F)$ of the form

$$T^4 + aT^3 + bpT^2 + ap^3T + p^6 \in \mathbb{Z}[T]\,,$$

where

$$a = a_p(t) = Tr(F), \;\; b = b_p(t) = (Tr(F^2) - Tr(F)^2)/2p.$$

from which we get the local Euler factor

$$1 + ap^{-s} + bp^{1-2s} + ap^{3-3s} + p^{6-4s}$$

for the $L$-function of $H^3(Y_t)$.

4.1. **Unit root method.** Let us suppose that the Frobenius polynomial is irreducible, but factors over $\mathbb{Z}_p$ as

$$(T-u)(T-v)(T-p^3/v)(T-p^3/u) \in \mathbb{Z}_p[T]$$

with $ord_p(u)=0, ord_p(v)=1$. Then $u$ is called the *unit-root* and according to DWORK [16], this unit root $u=u(t)$ can be computed from the holomorphic solution $\phi_0(t)$ using $p$-adic analytic continuation of

$$\frac{\phi_0(t)}{\phi_0(t^p)}$$

and evaluation at Teichmüller lift $\tilde{t}$ of $t \in \mathbb{P}^1$ (avoiding singular and supersingular values of $t$.) Dwork's unit-root method has been clarified by N. KATZ [20] by formulating it in terms of *crystals*. In her thesis, K. SAMOL [27] used this method to compute Euler factors for many families of Calabi-Yau varieties, using only the Picard-Fuchs equation. One of the important discoveries she made was that in many cases the method even worked at the singular points of the differential equation, and thus managed to determine weight four forms attached to $C$-points of Calabi-Yau operators, [28]. The explicit control of the $p$-adic analytic continuation can sometimes be obtained from *Dwork congruences* on the coefficients $A_n$ of the holomorphic solution. In the context of Calabi-Yau varieties defined by Laurent polynomials such *Dwork congruences* can be shown to hold [29], [23].

4.2. **Deformation method.** The type of crystals we are considering are defined over a ring $R$, which is a certain two-dimensional regular local sub-ring of $\mathbb{Z}_p[[t]]$. On $R$ there are two operations: the derivation

$$\Theta : R \longrightarrow R, \quad a \mapsto t\frac{\partial a}{\partial t}$$

and the lifted Frobenius map

$$\sigma : R \longrightarrow R, \quad a(t) \mapsto a(t^p)\,.$$

One has

$$\Theta \circ \sigma = p\,\sigma \circ \Theta\,.$$

We will consider a free $R$-module of rank four $H$, a non-degenerate symplectic pairing

$$\langle -,- \rangle : H \times H \longrightarrow R$$

and two operations

$$\nabla : H \longrightarrow H, \quad F : H \longrightarrow H$$

that we call the *Gauss-Manin* and *Frobenius*. The operator $\nabla$ a connection, so is supposed to satify the appropriate Leibniz rule, whereas $F$ is $\sigma$-linear. These three structures are required to satisfy the following compatibilities

i) $\Theta\langle x,y\rangle = \langle \nabla x, y\rangle + \langle x, \nabla y\rangle$.
ii) $p^3\langle x,y\rangle = \langle Fx, Fy\rangle$.
iii) $\nabla F = p\,F\nabla$.

Furthermore, we will have a Hodge-filtration

$$Fil^3 \subset Fil^2 \subset Fil^1 \subset Fil^0 = H$$

with

$$\nabla(Fil^i) \subset Fil^{i-1}, \quad F(Fil^i) \subset p^i H\,.$$

The first part of the structure may be called a *polarised F-crystal*, including the filtration makes us speak about a *polarised divisible Hodge F-crystals* (Fontaine-Lafaille crystals), we will call it a *CY-crystal* for short. Let us try to associate such a structure to a differential operator of the form

$$\mathcal{P} := \Theta^4 + tP_1(\Theta) + t^2P_2(\Theta) + \ldots + t^rP_r(\Theta)\,.$$

For this, we write everything out in MATRIX-form. We let

$$H := \sum_{i=0}^{3} R\phi_i\,,$$

where the $\phi_i$ are abstract basis vectors, that behave with respect to differentiation as the Frobenius basis of $\mathcal{P}$. Writing out the action of $\Theta$ on them, we can construct the companion matrix $A(t)$ for the connection $\nabla$ on $H$ corresponding to $\mathcal{P}$:

$$\nabla = t\frac{d}{dt} - A(t)\,,$$

where $A(t)$ is of the form

$$A(t) = \begin{pmatrix} 0 & 0 & 0 & * \\ 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} = A_0 + A_1t + A_2t^2 + \ldots + A_rt^r \in \mathbb{Q}[t]^{4\times 4}\,.$$

Because of the MUM-condition, we have

$$A_0 = N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrix $\Sigma$ of the symplectic form at $t=0$ can be taken to be of the form

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

We now write the Frobenius matrix in a series

$$F = F(t) = F_0 + F_1t + F_2t^2 + \ldots$$

The above conditions, especially the Griffiths transversality and divisibility, lead to a very specific form for the constant term $F_0$:

$$F_0 = \begin{pmatrix} \xi & 0 & 0 & 0 \\ p\alpha & \xi p & 0 & 0 \\ p^2\beta & p^2\alpha & \xi p^2 & 0 \\ p^3\gamma & p^3\beta & p^3\alpha & \xi p^3 \end{pmatrix},$$

where $\xi^2 = 1$ and $\xi\beta = \alpha^2/2$. One give an explicit formula for the series $F(t)$ as

$$F(t) = E(t^p)^{-1}F_0E(t) \in \mathbb{Q}[t]^{4\times 4}\,,$$

where the matrix $E(t)$ is a modification of the *fundamental matrix* for the differential equation

$$\tilde{E}_{jk} = \Theta^k \phi_j = \begin{pmatrix} \phi_0 & \Theta(\phi_0) & \Theta^2(\phi_0) & \Theta^3(\phi_0) \\ \phi_1 & \Theta(\phi_1) & \Theta^2(\phi_1) & \Theta^3(\phi_1) \\ \phi_2 & \Theta(\phi_2) & \Theta^2(\phi_2) & \Theta^3(\phi_2) \\ \phi_3 & \Theta(\phi_3) & \Theta^2(\phi_3) & \Theta^3(\phi_3) \end{pmatrix} \in \mathbb{Q}[[t]][\log t]^{4\times 4}\,.$$

This matrix reduces mod $t$ to $E^0_{jk} := \Theta^k \log^j(t)/j!$ and we set

$$E := (E^0)^{-1}\tilde{E} = {}''\tilde{E}\Big|_{\log(t)=0}{}''.$$

In all examples we have computed so far, we could make the following

***Observations***

- All terms of the series $F(t)$ are $p$-adically integral (depending linearly on $\alpha, \beta, \gamma$.)
- One can write
$$F(t) = \frac{\varphi(t)}{\Delta(t)^{p-1}} \bmod p^3\,,$$
where $\varphi(t) \in (\mathbb{Z}/p^3)[t]^{4\times 4}$ is a polynomial matrix and $\Delta(t)$ is the discriminant of the operator $\mathcal{P}$.
- The poles cancel at all singularities of $\mathcal{P}$, except for the apparent singularities. So if $\mathcal{P}$ does not have apparent singularities, the matrix $F(t) \mod p^3$ is in fact polynomial.
- We can 'trivially' read off
$$a(t) = -TrF(t) \mod p^3\,, \quad b(t) = (Tr(F(t)^2) - Tr(F(t))^2)/2p \mod p^3$$
and these do not depend on the choice of $\alpha, \beta, \gamma$ (this was already observed in [27].) This suffices to determine the local Euler factor at $p$ for $p \geq 5$.

Using this, we can compute Euler factors even at the singular points, as long as they are not apparent singularities. In particular, it works at the conifold points and we do not have to care about super-singular behaviour. For example, for the above mentioned operator **AESZ 34** one finds characteristic polynomials of Frobenius of the form

$$T(T - p\chi(p))(T^2 - a_pT + p^3)$$

for some character $\chi$. We find

| | $1/25$ | $1/9$ | $1$ |
|---|---|---|---|
| $a_7$ | $32$ | $-16$ | $-16$ |
| $a_{11}$ | $-60$ | $12$ | $12$ |
| $a_{13}$ | $-34$ | $38$ | $38$ |
| $a_{17}$ | $42$ | $-126$ | $-126$ |

So we recognise, using the table in [22], the weight four cusp forms $6/1$ for $\Gamma_0(6)$ at $t = 1$ and $t = 1/9$, and the form $30/1$ for $\Gamma_0(30)$ at $t = 1/25$.

4.3. **Lifting to higher order.** Let us set $\alpha = \beta = 0$ and $\xi = 1$, but keep $\gamma$ as a parameter. It appears that there is a *unique choice* for $\gamma$ mod $p$ for which

$$F(t) = \frac{\varphi(t)_{m(p)}}{\Delta(t)^{\ell(p)}} \bmod p^4$$

where $\ell(p)$ is a small slope linear functions of $p$ and $\varphi(t)_{m(p)}$ is a matrix-polynomial of small degree $m(p)$ linear in $p$. For all other choices of $\gamma$ this structure seems to get lost. By playing the same game modulo $p^5, p^6, p^7$, etc, we can determine a number $\gamma$ modulo $p^2, p^3$, etc. Continuing this way, we obtain a well-defined $p$-adic number $\gamma$ that goes into the Frobenius matrix at the MUM-point:

$$F_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ \gamma & 0 & 0 & p^3 \end{pmatrix}$$

For the quintic and $p = 11$ one finds

$$\gamma = 2 + 2 \cdot 11 + 3 \cdot 11^2 + 7 \cdot 11^3 + 5 \cdot 11^4 + 5 \cdot 11^5 + 6 \cdot 11^6 + \dots$$

Recall the relation between the $p$-adic $\zeta(3)$ and the $p$-adic gamma function:

$$-2\zeta_p(3) = \log \Gamma_p'''(0) = \Gamma_p'''(0) - \Gamma_p'(0)^3 .$$

The following marvellous miracle seems to take place:

***Observation***

- $\gamma = r \cdot \zeta_p(3)$.
- $r = c_3(X)/d$, where $d$ is the degree of the mirror manifold.

For the quintic $r = 200/5 = 40$. This is reminiscent of a very similar matrix describing the hermitian form $\langle x, \overline{y} \rangle$, where $\bar{\ }$ is the Frobenius at $\infty$, that is, complex conjugation, and the real $\zeta(3)$ appears at the place of $\zeta_p(3)$!

This is the end of the talk and of the conference, but I feel it is the beginning of something great.
During the conference we have seen some *amajzing maths*, we had a *great taam*, it was really a *naas workshop*.

***A great thank to the organisers Masha, Ling and Wadim!***

## References

[1] G. Almkvist, *The art of finding Calabi–Yau differential equations*, Dedicated to the 90-th birthday of Lars Gårding. Gems in Experimental Mathematics, 1 18, Contemp. Math. **517**, Amer. Math. Soc., Providence, RI, 2010.
[2] G. Almkvist, C. van Enckevort, D. van Straten, W. Zudilin, *Tables of Calabi–Yau operators*, `arXive:math/0507430`(2010).
[3] G. Almkvist, D. van Straten, *Update on Calabi–Yau operators*, in preparation.
[4] G. Almkvist, W. Zudilin, *Differential equations, mirror maps and zeta values*, in: Mirror symmetry. V, 481515, AMS/IP Stud. Adv. Math., **38**, Amer. Math. Soc., Providence, RI, (2006).
[5] Y. André, *G-functions and geometry*, Aspects of Mathematics, E13. Friedr. Vieweg & Sohn, Braunschweig (1989).
[6] M. Bogner, *On differential operators of Calabi–Yau type*, Thesis, Mainz (2012).

[7] M. Bogner, *Algebraic characterization of Differential Operators of Calabi–Yau type*, `ArXive:math.AG 1304.5434`.

[8] P. Candelas, X. de la Ossa, P. Green, L. Parkes, *An exactly soluble superconformal theory from a mirror pair of Calabi–Yau manifolds*, Phys. Lett. B 258 (1991), no. 1-2, 118 - 126.

[9] P. Candelas, X. de la Ossa, D. van Straten, *Local Euler factors from Picard Fuchs equations*, in preparation.

[10] Calabi–Yau Database, Version 2.0 (url: www2.mathematik.uni-mainz.de/CYequations/db/), Version 3.0 (url: cydb.mathematik.uni-mainz.de).

[11] D. Cox, S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs, **68**. American Mathematical Society, Providence, RI, (1999).

[12] P. Deligne, *Local behavior of Hodge structures at infinity*, Mirror Symmetry II, Studies in advanced mathematics, vol. 1, AMS/IP, 1997, 683 - 699.

[13] E. Delaygue, *A criterion for the integrality of the Taylor coefficients of mirror maps in several variables*, Adv. Math. **234** (2013), 414 - 452.

[14] L. Dieulefait, *On the modularity of rigid Calabi–Yau threefolds: epilogue.* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 377 (2010), Issledovaniya po Teorii Chisel. 10, 44–49, 241; translation in J. Math. Sci. (N.Y.) 171 (2010), no. 6, 725727.

[15] B. Dwork, *On the zeta function of a hypersurface, III*, Ann. of Math. **(2)** 83 (1966) 457 - 519.

[16] B. Dwork, *p-adic cycles. Inst. Hautes tudes Sci. Publ. Math. No. 37 (1969), 27115.*

[17] J. Gray, *Fuchs and the theory of differential equations*, Bulletin of the AMS (New series), Vol. **10** (1984), 1 - 26.

[18] F. Gouvêa, N. Yui, *Rigid Calabi–Yau threefolds over* $\mathbb{Q}$ *are modular.* Expo. Math. 29 (2011), no. 1, 142 - 149.

[19] E. Ince, *Ordinary Differential Equations*, Longmans, Green and Co. Ltd., New York, Toronto, Calcutta, Madras (1927).

[20] N. Katz, *Travaux de Dwork* (French) Sminaire Bourbaki, 24me anne (1971/1972), Exp. No. 409, pp. 167200. Lecture Notes in Math., Vol. 317, Springer, Berlin, 1973.

[21] C. Krattenthaler, *On the integrality of Taylor coefficients of mirror maps*, Duke Math. J. **151** (2010), 175 - 218.

[22] C. Meyer, *Modular Calabi–Yau threefolds.* Fields Institute Monographs **22**, American Mathematical Society, Providence, RI, (2005).

[23] A. Mellit, M. Vlasenko, *Dwork's congruences for the constant terms of powers of a Laurent polynomial*, Int. J. Number Theory 12 (2016), no. 2, 313 - 321.

[24] D. R. Morrison, *Geometric aspects of mirror symmetry*, Mathematics unlimited 2001 and beyond, 899 - 918, Springer, Berlin, (2001).

[25] C. Peters, J. Steenbrink, *Mixed Hodge Structures*, Ergebnisse der Mathematik und ihre Grenzgebiete, 3. Folge, Springer Verlag, (2008).

[26] C. Schoen, *On the geometry of a special determinantal hypersurface associated to the Mumford-Horrocks vector bundle.* J. Reine Angew. Math. 364 (1986), 85111.

[27] K. Samol, *Frobenius Polynomial for Calabi–Yau Equations*, Thesis, Mainz (2010).

[28] K. Samol and D. van Straten, *Frobenius polynomials for Calabi–Yau equations*, Commun. Number Theory Phys. **2**, no. 3, 537 - 561, (2008).

[29] K. Samol, D. van Straten, *Dwork congruences and reflexive polytopes*, Ann. Math. Quebec **39**, (2015), 185 - 203.

[30] J. Steenbrink, *Limits of Hodge Structures*, Inv. Math **31**,(1976), 229 - 257.

[31] D. van Straten,*Calabi–Yau operators*, `ArXiv:1704.00164 [math.AG]`.

[32] H. Verrill, *Root lattices and pencils of varieties.* J. Math. Kyoto Univ. 36 (1996), no. 2, 423 - 446.

[33] C. Voisin, *Symmetrie mirroir*, Panorama et synthése, Soc. Math. France, translated as: *Mirror symmetry*, AMS, Providence, RI (1999).

[34] V. Vologodsky, *On the N-Integrality of instanton numbers*, `arXive:0707.4617 [math.AG]` (2008).

[35] A. Wiles, *Modular elliptic curves and Fermat's last theorem.* Ann. of Math. (2) 141 (1995), no. 3, 443551.

[36] A. Wiles, *Modular forms, elliptic curves, and Fermat's last theorem.* Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zrich, 1994), 243245, Birkhuser, Basel, 1995.

[37] N. Yui, *Modularity of Calabi–Yau varieties: 2011 and beyond.* Arithmetic and geometry of K3 surfaces and Calabi–Yau threefolds, 101139, Fields Inst. Commun., 67, Springer, New York, 2013.